

Asymptotic behavior of global entropy solutions for nonstrictly hyperbolic systems with linear damping

Richard A. De la Cruz
Juan C. Juajibioy
Leonardo Rendón
Bogotá

2014

Abstract

In this paper we investigate the large time behavior of the global weak entropy solutions to the symmetric Keyfitz-Kranzer system with linear damping. It is proved that as $t \rightarrow \infty$ the entropy solutions tend to zero in the L^p norm

1 Introduction

In this paper we consider the Cauchy problem to the symmetric system of Keyfitz-Kranzer type with linear damping

$$\begin{cases} u_t + (u\phi(r))_x + au = 0, \\ v_t + (v\phi(r))_x + bv = 0. \end{cases} \quad (1.1)$$

with initial data

$$u(x, 0) = u_0(x), \quad v(x, 0) = u_0(x), \quad (1.2)$$

This system models of propagation of forward longitudinal and transverse waves of elastic string which moves in a plane, see [1], [3]. General source term for the system (2.8) was considered in [6]. The damping in the system (2.8) represents external forces proportional to velocity, and this term can be produce lost of total energy of system. Consider the scalar case, by example

$$u_t + au_x + bu = 0, \quad u(x, 0) = u_0(x). \quad (1.3)$$

From the integral representation of (1.3) it is easy to find the following solution

$$u(x, t) = u_0(x - at)e^{-bt}. \quad (1.4)$$

The term bu produce a dissipative effect in the solutions, i.e, the solutions tends to zero when $t \rightarrow \infty$. We are looking for condition under wich the terms a, b have a dissipative effect in the solutions of 2.8.

Let $r(x, t) = \sqrt{u(x, t)^2 + v(x, t)^2}$ be, we are going to show the following main theorem.

Theorem 1.1. *If the initial data $(u_0(x), v_0(x)) \in L^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$ then the Cauchy problem (2.8), (1.2) has a weak entropy solutions satisfying*

$$\|u\|_{L^\infty(\Omega)} + \|u\|_{L^\infty(\Omega)} < M \quad (1.5)$$

Moreover $r(u, v)$ converges to zero in L^p with exponential time decay, i.e.

$$\|r(x, t)\|_{L^p(\mathbb{R})} \leq K e^{-Mt} \|r(x, 0)\|_{L^p(\mathbb{R})} \quad (1.6)$$

2 Preliminars

We start with some preliminaries about the general systems of conservation laws, see [2] chapter 5. Let $f : \Omega \rightarrow \mathbb{R}^n$ be a smooth vector field. Consider Cauchy problem for the system

$$\begin{cases} u_t + f(u)_x = g(u), \\ u(x, 0) = u_0(x). \end{cases} \quad (2.1)$$

When $g(u) = 0$ the system (2.1) is called homogeneous system of conservation laws, if $g(u) \neq 0$ the system (2.1) is called inhomogeneous system or balance system of consevation laws. We shall work also with the parabolic perturbation to the system (2.1), namely

$$\begin{cases} u_t + f(u)_x = \epsilon u_{xx} + g(u), \\ u(x, 0) = u_0(x). \end{cases} \quad (2.2)$$

Denote by $A(u) = Df(u)$ the Jacobian matrix of partial derivates of f .

Definition 2.1. The system (2.1) is strictly hyperbolic if for every $u \in \Omega$, the matrix $A(u)$ has n real distinct eigenvalues $\lambda_1(u) < \dots < \lambda_n(u)$.

Let $r_i(u)$ the correspond eigenvetor to $\lambda_i(u)$, then

Definition 2.2. We say that the i -th characteristic field is genuinely non-linear if

$$\nabla \lambda_i(u) \cdot r_i(0) \neq 0, \quad (2.3)$$

If instead

$$\nabla \lambda_i(u) \cdot r_i(0) = 0, \quad (2.4)$$

we say that the i -th characteristic field is linearly degenerate.

For the following definitions see [5], [7]

Definition 2.3. A k -Riemann invariant is a smooth function $w_k : \mathbb{R}^n \rightarrow \mathbb{R}$, such that

$$\nabla w_k(u) \cdot r_k(u) = 0 \quad (2.5)$$

Definition 2.4. A pair of function $\eta, q : \mathbb{R}^n \rightarrow \mathbb{R}$ is called a entropy-entropy flux pair if it satisfies

$$\nabla \eta(u) A(u) = \nabla q(u), \quad (2.6)$$

if $\eta(u)$ is a convex function then the pair (η, q) is called convex entropy-entropy flux pair.

Definition 2.5. A bounded measurable function $u(x, t)$ is an entropy (or admissible) solution for the Cauchy problem (2.1), if it satisfies the following inequality

$$\eta(u)_t + q(u)_x + \nabla \eta(u) g(u) \leq 0. \quad (2.7)$$

in the distributional sense, where (η, q) is any convex entropy-entropy flux pair.

We consider the general system of Keyfitz-Kranzer system

$$\begin{cases} u_t + (u\phi(u, v))_x = 0, \\ v_t + (v\phi(u, v))_x = 0, \end{cases} \quad (2.8)$$

to get some general observations about this type of systems. Making $F(u, v) = (u\phi(u, v), v\phi(u, v))$ in (2.8), we have that the eigenvalues and eigenvector of the Jacobian's matrix Df are given by

$$\lambda_1(u, v) = \phi(u, v) \quad r_1 = (1, -\frac{\phi_u}{\phi_v}) \quad (2.9)$$

$$\lambda_2(u, v) = \phi(u, v) + (u, v) \cdot \nabla \phi(u, v) \quad r_2 = (1, \frac{v}{u}). \quad (2.10)$$

From (2.9), (2.10) we have that $\nabla \phi \cdot r_1 = 0$, and $\nabla Z(u, v) \cdot r_2 = 0$, where $Z(u, v) = \frac{u}{v}$, then the Riemann invariants are given by

$$W(u, v) = \phi(u, v), \quad (2.11)$$

$$Z(u, v) = \frac{u}{v}. \quad (2.12)$$

Lemma 2.6. *The system (2.8) is always linear degenerate in the first characteristic field. If*

$$(u, v) \nabla \phi(u, v) \neq 0,$$

then the system (2.8) is strictly hyperbolic and non linear degenerate in the second characteristic field, moreover

$$\nabla \lambda_2(u, v) \cdot r_2 = \frac{2(u, v) \nabla \phi(u, v) + (u, v) H(\phi)(u, v)^T}{u} \quad (2.13)$$

where H represents the Hessian matrix.

Lemma 2.7. *Let $\eta(u, v) \in \mathbf{C}^1(\mathbb{R}_+)$ a Lipschitz function in a neighborhood of the origin, $q(u, v) = \psi(u, v) + \eta(u, v)\phi(u, v)$ be a function, such that ψ satisfies*

$$\nabla \psi(u, v) = ((u, v) \cdot \nabla \eta(u, v) - \eta(u, v)) \nabla \phi(u, v). \quad (2.14)$$

Then the pair

$$(n(u, v), q(u, v)) \quad (2.15)$$

is a entropy-entropy flux pair for the system (2.8). Moreover if $\eta(u, v)$ is a convex function, then the pair (2.15) is a convex entropy-entropy flux pair.

3 Global existence of weak entropy solutions and asymptotic behavior

We consider the parabolic regularization of the system (2.8), namely

$$\begin{cases} u_t + (u\phi(r))_x + au = \epsilon u_{xx}, \\ v_t + (v\phi(r))_x + bv = \epsilon v_{xx}, \end{cases} \quad (3.1)$$

with initial data

$$u^\epsilon(x, 0) = u_0^\epsilon * j_\epsilon, \quad v^\epsilon(x, 0) = v_0^\epsilon * j_\epsilon, \quad (3.2)$$

where j_ϵ is a mollifier. In this case $\phi(u, v) = \phi(r)$, with $r = \sqrt{u^2 + v^2}$. By (2.9) the eigenvectors and eigenvalues are given by

$$\lambda_1(u, v) = \phi(r) \quad r_1 = \left(1, -\frac{u}{v}\right) \quad (3.3)$$

$$\lambda_2(u, v) = \phi(r) + r\phi'(r) \quad r_2 = \left(1, \frac{v}{u}\right). \quad (3.4)$$

The following conditions will be necessary in our next discussion

$$C_1 \quad \lim_{r \rightarrow 0} r\phi(r) = 0, \quad r\phi'(r) \neq 0$$

$$C_2 \quad a > b$$

The condition C_1 guarantees the strictly hyperbolicity to the system (3.2), while condition C_2 ensures the existence of a positive invariant region. Now we consider the following subset of \mathbb{R}

$$\Sigma = \{(u, v) : \phi(r) \leq C_0, 0 < C_1 \leq \frac{u}{v} \leq C_2\}.$$

We affirm that Σ is an invariant region. Let $h(u, v) = (au, bv)$ be, if $(\bar{u}, \bar{v}) \in \gamma_1$ where γ_1 is the level curve of $Z = \phi(r)$ we have that

$$(\nabla W \cdot h)(\bar{u}, \bar{v}) = (a + b)r\phi'(r) > 0$$

and if $\bar{u}, \bar{v} \in \gamma_2$ where γ_2 is the level curve of Z we have that

$$(\nabla Z \cdot h)(\bar{u}, \bar{v}) = (a - b)\alpha_i > 0$$

with $i = 1, 2$, then by the Theorem 14.7 of [5], Σ is an invariant region for the system (3.1). It is easy to verify that (au, bv) satisfies the condition $H_1 \cdots H_5$ in [6], thus we have the following Lemma.

Proposition 3.1. *If $(u_0, v_0) \in \Sigma$ and the C conditions hold, then the Cauchy problem (3.1), (3.2) has a global weak entropy solution.*

Now for the global behavior of solutions, using ideas of the author in [4], we construct the following entropy-entropy flux pairs

$$n(r) = r^m, \quad m \leq 2.$$

From (2.14) we have

$$q(r) = (m-1) \int_0^r s^m \phi'(s) ds + r^m \phi(r),$$

Integrating by parts we have that

$$q(r) = (m-1) \int_0^r s^m \phi'(s) ds + r^m \phi(r),$$

integrating by parts we have

$$q(r) = m\phi(r) - m(m-1) \int_0^r s^{m-1} \phi(s) ds.$$

Let $M = \sup_{(u,v) \in [0, \|u\|_{L^\infty}] \times [0, \|u\|_{L^\infty}]} \{\phi(r)\}$, then we have that

$$|q(r)| \leq 2mMr^m. \quad (3.5)$$

Multiplying in (2.8) by $\nabla \eta$ we have that

$$\eta(r)_t + q(r)_x \leq -3mMr^m \quad (3.6)$$

Now we choose $h(x) \in C^2(\mathbb{R})$ a function such a $|h'(x)| \leq 1$, $|h''(x)| \leq 1$ and $h(x) = |x|$ for $|x| \geq 1$ and set $k(x) = e^{-h(x)}$, then $k'(x) \leq k(x)$. Multiplying by $k(x)$ in (3.6), and integrating over x we have

$$\frac{d}{dt} \int_{-\infty}^{\infty} \eta(r) g(x) \leq \int_{-\infty}^{\infty} q(r) k'(x) + -3mM \int_{-\infty}^{\infty} r^m dx \quad (3.7)$$

by the inequality (3.5) we have

$$\frac{d}{dt} \int_{-\infty}^{\infty} \eta(r) k(x) dx \leq -mM \int_{-\infty}^{\infty} r^m k(x) dx. \quad (3.8)$$

If $\psi(t) = \int_{-\infty}^{\infty} \eta(r) k(x) dx$ we have

$$\frac{d}{dt} \psi(t) \leq -mM \psi(t),$$

by Gronwall's inequality we have

$$\psi(t) \leq e^{-mMt} \psi(0).$$

Thus we have

$$\left(\int_{-\infty}^{\infty} r^m(t) k(x) dx \right)^{\frac{1}{m}} \leq e^{-Mt} \left(\int_{-\infty}^{\infty} r^m(0) k(x) dx \right)^{\frac{1}{m}} \quad (3.9)$$

Passing to limit $m \rightarrow \infty$ in (3.9) we have the inequality (1.5)

4 Acknowledgments

We would like to thanks to professor Laurent Gosse by his suggestions and review. To the professor Juan Galvis by his many valuable observation, and to the professor Yun-Guang Lu by his suggestion this problem.

References

- [1] Herbert C. Kranzer Barbara L. Keyfitz, *A system of non-strictly hyperbolic conservation laws arising in elasticity theory*, Archive for Rational Mechanics and Analysis **72** (1980), 219–241.
- [2] Alberto Bressan, *Hyperbolic systems of conservatin laws: The one-dimensional cauchy problem*, Oxford University Press, 2005.
- [3] N. D. Cristescu, *Dynamic plasticity*, Wolrd Scientific, 2007.
- [4] E. Yu. Panov, *On the theory of generalized entropy solutions of the cauchy problem for a class of non-strictly hyperbolic system of conservation laws*, Sbornik Mathematics **191** (1999), 127–157.
- [5] Joel Smoller, *Shock waves and reaction-diffusion equations*, Springer-Verlag, 1994.
- [6] Guo-Quian Song, *Existence of global weak solutions to a symmetrically hyperbolic system with a source*, Revista Colombiana de Matemáticas **42** (2008), 221–232.
- [7] Hui-Min Yu, *Large time behavior of entropy solutions to some hyperbolic sysystems with dissipative structure*, Acta Mathematicae Applicatae Sinica, English Series **29** (2013), no. 3, 509–515.